Journal of Multivariate Analysis **67**, 398–413 (1998)
Article No. MV981768

Weak Limits for Multivariate Random Sums

Tomasz J. Kozubowski*

The University of Tennessee at Chattanooga

and

Anna K. Panorska†

The University of Tennessee at Chattanooga and BlueCross BlueShield of Tennessee

Received July 24,1996; revised April 27, 1998

IN MEMORY OF PROFESSOR STAMATIS CAMBANIS

Let $\{\mathbf{X}_i, i \geqslant 1\}$ be a sequence of i.i.d. random vectors in R^d, and let ν_p, $0 < p < 1$, be a positive, integer valued random variable, independent of $\mathbf{X}_i$s. The ν-stable distributions are the weak limits of properly normalized random sums $\sum_{i=1}^{\nu_p} \mathbf{X}_i$, as $\nu_p \xrightarrow{P} \infty$ and $p\nu_p \xrightarrow{d} \nu$. We study the properties of ν-stable laws through their representation via stable laws. In particular, we estimate their tail probabilities and provide conditions for finiteness of their moments. © 1998 Academic Press

AMS 1991 subject classifications: primary 60E05; secondary 60E07, 60G50, 62H05.

Key words and phrases: random sum, stable law, heavy-tailed distribution, geometric stable distribution, Linnik distribution, tail probability, mixture.

1. INTRODUCTION AND NOTATION

The goal of this paper is the introduction of a new class of multivariate distributions, the ν-stable laws, that includes stable and geometric stable laws as special cases. Let $\{\mathbf{X}_i, i \geqslant 1\}$ be a sequence of i.i.d. random vectors (r.v.) in R^d. Consider a random sum

$$\mathbf{S}_{\nu_p} = \mathbf{X}_1 + \cdots + \mathbf{X}_{\nu_p}, \tag{1}$$

where ν_p, $0 < p < 1$, is an integer valued random variable (r.v.) and $\nu_p \xrightarrow{P} \infty$ (in probability) while $p\nu_p \xrightarrow{d} \nu$ as $p \to 0$ (in distribution). The

* The research of this author was partially supported by the UC Foundation grant from the University of Tennessee at Chattanooga.

† The research of this author was partially supported by the UC Foundation Summer Fellowship from the University of Tennessee at Chattanooga.

0047-259X/98 $25.00

limiting distribution of normalized $\mathbf{S}_{\nu_p}$ is called *ν-stable*. If the sum (1) is deterministic, then the limiting distributions are stable laws (see [20] and references therein). If ν_p is a geometric random variable with mean $1/p$ then the limit is a geometric stable (GS) law (see [1, 5, 8, 11, 15, 16, 18]).

Random summation appears in applied problems in many fields, including physics, biology, economics, insurance mathematics, reliability and queuing theories (see, [7] and the references therein). Since the ν-stable distributions approximate random sums, they have many natural applications in stochastic modeling. In particular, GS laws successfully compete with stable laws in modeling financial asset returns (see, e.g., [2, 12, 14, 16, 17]).

We show that the representation, moments, and tail behavior of univariate ν-stable distributions, as discussed in [13], naturally extend to the multivariate case. We prove that stable and ν-stable vectors have similar tail behavior. However, as established in [18], they may have very different asymptotics at the origin (for example, densities of GS laws may be unbounded at zero; in fact sharp peaks at the origin coupled with heavy tails are the features of GS distributions that make them particularly useful in financial modeling).

The paper is organized as follows. In Section 2 we define multivariate ν-stable distributions and give examples. In Section 3, we establish a representation of ν-stable r.v.'s as location and scale mixtures of stable laws. This representation provides the most effective tool in the further study of ν-stable vectors. We also discuss some fundamental properties of ν-stable distributions. The rest of the paper focuses on tail probabilities in Section 4 and moments of ν-stable distributions in Section 5.

Notation. We use the following notation throughout the paper. For any $\mathbf{s} = (s_1, ..., s_d)$ and $\mathbf{t} = (t_1, ..., t_d)$ from R^d, we let $\langle \mathbf{s}, \mathbf{t} \rangle = \sum_{i=1}^d t_i s_i$ and $\|\mathbf{t}\| = \langle \mathbf{t}, \mathbf{t} \rangle^{1/2} = (\sum_{i=1}^d t_i^2)^{1/2}$ denote the inner product and the corresponding norm in R^d. The unit sphere in R^d is denoted as $S_d = \{\mathbf{s} \in R^d \colon \|\mathbf{s}\| = 1\}$. Further, "$\xrightarrow{d}$" denotes weak convergence, while "$\xrightarrow{P}$" represents convergence in probability.

2. DEFINITIONS

Let $\{\nu_p, p \in (0, 1)\}$ be a family of non-negative, integer-valued random variables, such that when $p \to 0$,

$$\nu_p \xrightarrow{P} \infty \qquad \text{and} \qquad p\nu_p \xrightarrow{d} \nu, \tag{2}$$

where ν is a non-negative random variable with distribution function (d.f.) A ($\nu \sim A$).

Under the above conditions, we define ν-stable random vectors and distributions as follows.

DEFINITION 2.1. A random vector $\mathbf{Y}$ (and its distribution) is said to be ν-stable, if there exists an independent of ν_p sequence of i.i.d. random vectors $\mathbf{X}_1, \mathbf{X}_2, \ldots$, and $a = a(p) > 0$, $\mathbf{b} = \mathbf{b}(p) \in R^d$ such that

$$a(p) \sum_{i=1}^{\nu_p} (\mathbf{X}_i + \mathbf{b}(p)) \xrightarrow{d} Y, \qquad \text{as} \quad p \to 0. \tag{3}$$

We say that a ν-stable r.v. $\mathbf{Y}$ is *regular* if γ, the Laplace transform of ν, satisfies the condition:

$$\gamma(-\log \mathbf{\Phi}_1(\mathbf{t})) = \gamma(-\log \mathbf{\Phi}_2(\mathbf{t})) \qquad \text{for all } \mathbf{t} \text{ implies that } \mathbf{\Phi}_1 \equiv \mathbf{\Phi}_2,$$

where $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ are infinitely divisible characteristic functions. In the sequel, we restrict our attention to regular ν-stable laws. The classical Transfer Theorem and its converse (see, e.g., [19, 21]) lead to the following characterization of regular ν-stable random vectors.

PROPOSITION 2.1. *A random vector $\mathbf{Y} = (Y_1, \ldots, Y_d)$ is regular ν-stable if and only if its ch.f. $\mathbf{\Psi}(\mathbf{t}) = E\exp\{i\langle \mathbf{t}, \mathbf{Y}\rangle\}$ has the form*

$$\mathbf{\Psi}(\mathbf{t}) = \gamma(-\log \mathbf{\Phi}(\mathbf{t})), \tag{4}$$

where γ is the Laplace transform of ν and $\mathbf{\Phi}$ is the ch.f. of a stable law in R^d.

Proof. The proof is analogous to that of the one-dimensional case as presented in [13], and thus it is omitted. ∎

In view of (4) and the spectral representation of stable laws (see, e.g., [20]), there exist a parameter α $(0 < \alpha \leqslant 2)$, a finite measure $\mathbf{\Gamma}$ on S_d, and a vector $\mathbf{m} \in R^d$, such that

$$\mathbf{\Psi}(\mathbf{t}) = \gamma\left(\int_{S_d} |\langle \mathbf{t}, \mathbf{s}\rangle|^\alpha \, \omega_{\alpha,1}(\langle \mathbf{t}, \mathbf{s}\rangle)\, \mathbf{\Gamma}(d\mathbf{s}) - i\langle \mathbf{t}, \mathbf{m}\rangle\right), \tag{5}$$

where

$$\omega_{\alpha,\beta}(x) = \begin{cases} 1 - i\beta \operatorname{sign}(x) \tan(\pi\alpha/2), & \text{if} \quad \alpha \neq 1, \\ 1 + i\beta \dfrac{2}{\pi} \operatorname{sign}(x) \log|x|, & \text{if} \quad \alpha = 1. \end{cases} \tag{6}$$

We say that the r.v. $\mathbf{Y} = (Y_1, \ldots, Y_d)$ given by (5)–(6) is $\nu - stable$ (and its components $Y_1, \ldots, Y_d$ are *jointly ν-stable*) with *spectral representation* $(\mathbf{\Gamma}, \mathbf{m})$, where $\mathbf{\Gamma}$ is the *spectral measure*. To emphasize the *shape* parameter α, we use the notation: $\mathbf{Y} \sim \nu_\alpha(\mathbf{\Gamma}, \mathbf{m})$. Similarly, a stable r.v. $\mathbf{X}$ with spectral measure $\mathbf{\Gamma}$ and location parameter $\mathbf{m}$ (with the ch.f. $\mathbf{\Phi}$ as in (4)), will be denoted as $\mathbf{X} \sim S_\alpha(\mathbf{\Gamma}, \mathbf{m})$. If $\mathbf{\Phi}$ corresponds to a strictly stable distribution in R^d, we will say that $\mathbf{Y}$ is *strictly* ν-stable. The conditions (in terms of $\mathbf{\Gamma}$ and $\mathbf{m}$) for strict stability of $\mathbf{X}$ (see [20]) lead to the following alternative definition of strictly ν-stable $\mathbf{Y}$.

DEFINITION 2.2. A random vector $\mathbf{Y}$ has a strictly ν-stable distribution in R^d if its ch.f. follows the representation (5) and (6) with either $\mathbf{\Gamma} \equiv \mathbf{0}$ or $\mathbf{\Gamma} \not\equiv \mathbf{0}$ and

$$\begin{cases} \mathbf{m} = \mathbf{0}, & \text{if } \alpha \neq 1, \\ \int_{S_d} s_k \mathbf{\Gamma}(d\mathbf{s}) = 0, \quad \text{for} \quad k = 1, ..., d, & \text{if } \alpha = 1. \end{cases} \tag{7}$$

A random vector $\mathbf{Y} \sim \nu_\alpha(\mathbf{\Gamma}, \mathbf{m})$ is symmetric ν-stable if and only if $\alpha = 2$ and $\mathbf{m} = \mathbf{0}$, or $0 < \alpha < 2$, $\mathbf{m} = \mathbf{0}$, and $\mathbf{\Gamma}$ is a finite, symmetric measure on S_d, in which case

$$\mathbf{\Psi}(\mathbf{t}) = \gamma\left(\int_{S_d} |\langle \mathbf{t}, \mathbf{s} \rangle|^\alpha \mathbf{\Gamma}(d\mathbf{s})\right).$$

REMARKS AND EXAMPLES. 1. In the one dimensional case (5) reduces to

$$\psi(t) = \gamma(\sigma^\alpha |t|^\alpha \omega_{\alpha, \beta}(t) - i\mu t), \tag{8}$$

the ch.f. of a ν-stable random variable $\nu_\alpha(\sigma, \beta, \mu)$ (see [13]).

2. All constant random vectors in R^d are ν-stable.

3. If ν_p has a Poisson distribution with mean $1/p$, then $\nu \equiv 1$ and (5) reduces to ch.f. a stable law in R^d (see [20]). Thus, stable distributions are regular ν-stable.

4. If ν_p is a geometric random variable with mean $1/p$, then ν has a standard exponential distribution with $\gamma(z) = (1+z)^{-1}$, and (5) reduces to the ch.f. of a geometric stable distribution (see, e.g., [1, 8, 11, 15, 16, 18]). Thus, GS distributions are regular ν-stable. A more general class of *generalized Linnik distributions* is obtained when ν_p has a negative binomial distribution, in which case ν has a Gamma distribution (see [5] for the univariate case).

5. Both stable and (strictly) GS laws have the *stability property*. In the GS case, for any $0<p<1$ there exists $a(p)>0$, such that

$$a(p) \sum_{i=1}^{\nu_p} \mathbf{X}_i \stackrel{d}{=} \mathbf{X}_1 \text{ (in distribution)}, \tag{9}$$

where ν_p is geometrically distributed random variable with mean $1/p$, independent of $\{\mathbf{X}_i, i \geqslant 1\}$ (see [11]). Equation (9) with other integer-valued random variables ν_p was studied in [4], [9], and [10]. Its solutions were called ν_p-stable laws. However, since (9) has nontrivial solutions only under very tight conditions on ν_p (see [10]), the class of ν_p-stable laws is quite restrictive. Our definition leads to a broader class of distributions.

6. In view of (4), all regular ν-stable r.v.'s can be interpreted as the values of a subordinated stable Lévy process. More precisely, if $\mathbf{Y} \sim \nu_\alpha(\mathbf{\Gamma}, \mathbf{m})$, then $\mathbf{Y} \stackrel{d}{=} \mathbf{X}(\nu)$, where $\mathbf{X}$ is a d-dimensional stable process with independent increments, $\mathbf{X}(0)=\mathbf{0}$, and $\mathbf{X}(1) \sim S_\alpha(\mathbf{\Gamma}, \mathbf{m})$. Consequently, ν-stable distributions may be studied via the theory of (stopped) Lévy processes (see [3]). Although we do not adopt this approach, let us note here that infinite divisibility of $\mathbf{Y}$ easily follows from (4), provided that the distribution of ν is infinitely divisible. One can then derive the Lévy measure of $\mathbf{Y}$ from that of $\mathbf{X}(1)$ (see, e.g., Lemma 7, VI.2, of [3], and also [6]). The two Lévy measures may have entirely different asymptotic behavior (for example, [15] shows that the rate at which the mass increases at the origin is logarithmic in the GS case, compared with a polynomial rate in the stable case).

3. REPRESENTATION AND PROPERTIES

In this section we derive a representation of ν-stable random vectors via stable distributions and establish some of their fundamental properties. We find that many properties of ν-stable distributions are similar to those of stable laws, and essentially do not depend on the particular distribution of ν.

The following result extends the representation of univariate ν-stable random variables (derived in [13]) to the multivariate case.

THEOREM 3.1. $\mathbf{Y} \sim \nu_\alpha(\mathbf{\Gamma}, \mathbf{m})$ *if and only if*

$$\mathbf{Y} \stackrel{d}{=} \begin{cases} \mathbf{m}\nu + \nu^{1/\alpha}\mathbf{X}, & \text{if} \quad \alpha \neq 1, \\ \mathbf{m}\nu + \left(\dfrac{2}{\pi}\nu \log \nu\right)\mathbf{g} + \nu\mathbf{X}, & \text{if} \quad \alpha = 1, \end{cases} \tag{10}$$

with

$$\mathbf{g} = (g_1, ..., g_d) = \int_{S_d} \mathbf{s}\boldsymbol{\Gamma}(d\mathbf{s}), \tag{11}$$

where $\mathbf{X} \sim S_\alpha(\boldsymbol{\Gamma}, \mathbf{0})$, $v \sim A$, *and* $\mathbf{X}$ *and* v *are independent.*

Proof. The proof is analogous to that of the one-dimensional case, as presented in [13]. Let $\boldsymbol{\Phi}$ be the ch.f. of $X \sim S_\alpha(\boldsymbol{\Gamma}, \mathbf{0})$. Note that for any $\mathbf{t} \in R^d$ and $z > 0$, we have

$$\boldsymbol{\Phi}(z^{1/\alpha}\mathbf{t}) = \begin{cases} [\boldsymbol{\Phi}(\mathbf{t})]^z \exp\left\{-\frac{2}{\pi} z \log z i\langle \mathbf{t}, \mathbf{g}\rangle\right\}, & \text{if} \quad \alpha = 1, \\ [\boldsymbol{\Phi}(\mathbf{t})]^z, & \text{if} \quad \alpha \neq 1. \end{cases}$$

Thus, the ch.f. of the RHS of (10) equals $E[\boldsymbol{\Phi}(\mathbf{t}) \exp\{i\langle \mathbf{t}, \mathbf{m}\rangle\}]^v$, which is the same as (5), since $Eu^v = \gamma(-\log u)$. ∎

The representation given in Theorem 3.1 and conditioning on v produce a relation between the distribution functions and densities (if the distributions are non-singular) of v-stable and stable random vectors. Let $G_{\alpha, \boldsymbol{\Gamma}, \mathbf{m}}(\cdot)$ and $F_{\alpha, \boldsymbol{\Gamma}}(\cdot)$ be d.f.'s of $v_\alpha(\boldsymbol{\Gamma}, \mathbf{m})$ and $S_\alpha(\boldsymbol{\Gamma}, \mathbf{0})$ r.v.'s respectively, and let $g_{\alpha, \boldsymbol{\Gamma}, \mathbf{m}}(\cdot)$ and $f_{\alpha, \boldsymbol{\Gamma}}(\cdot)$ be the corresponding densities.

COROLLARY 3.1. *The distribution function and density* (*if exists*) *of* $\mathbf{Y} \sim v_\alpha(\boldsymbol{\Gamma}, \mathbf{m})$ *can be expressed as*

$$G_{\alpha, \boldsymbol{\Gamma}, \mathbf{m}}(\mathbf{y}) = \int_0^\infty F_{\alpha, \boldsymbol{\Gamma}}(z^{-1/\alpha}\mathbf{y} - z^{1-1/\alpha}\mathbf{m} - \mathbf{c} \log z)\, A(dz), \tag{12}$$

$$g_{\alpha, \boldsymbol{\Gamma}, \mathbf{m}}(\mathbf{y}) = \int_0^\infty f_{\alpha, \boldsymbol{\Gamma}}(z^{-1/\alpha}\mathbf{y} - z^{1-1/\alpha}\mathbf{m} - \mathbf{c} \log z)\, z^{-d/\alpha}\, A(dz), \tag{13}$$

where $\mathbf{c} = (2/\pi)(1 - |sign(\alpha - 1)|)\, \mathbf{g}$ *with* $\mathbf{g}$ *as in* (11).

Theorem 3.1 and its Corollary show that v-stable distributions are location and scale mixtures of stable laws. The properties presented in the sequel follow from the corresponding properties of stable distributions via Theorem 3.1. First, we show that all multivariate marginals of a v-stable vector are v-stable.

PROPOSITION 3.1. *If* $\mathbf{Y} = (Y_1, ..., Y_d) \sim v_\alpha(\boldsymbol{\Gamma}, \mathbf{m})$, *then for all* $n \leqslant d$, $(Y_1, ..., Y_n) \sim v_\alpha(\boldsymbol{\Gamma}', \mathbf{m}')$, *where* $\boldsymbol{\Gamma}' = \hat{\boldsymbol{\Gamma}} \circ h^{-1}$, *and*

$$h: S'_d = \left\{(s_1, ..., s_d) \in S_d: \sum_{k=1}^{n} s_k^2 > 0\right\} \to S_n,$$

$$h(s_1, ..., s_d) = \left(\frac{s_1}{(\sum_{k=1}^n s_k^2)^{1/2}}, ..., \frac{s_n}{(\sum_{k=1}^n s_k^2)^{1/2}}\right),$$

$$\hat{\boldsymbol{\Gamma}}(d\mathbf{s}) = \left(\sum_{k=1}^{n} s_k^2\right)^{\alpha/2} \boldsymbol{\Gamma}(d\mathbf{s}), \qquad \mathbf{s} \in S'_d.$$

The parameter $\mathbf{m}' = (m'_1, ..., m'_n)$ *has a kth component* $(1 \leqslant k \leqslant n)$ *given by*

$$m'_k = \begin{cases} m_k, & \textit{if} \quad \alpha \neq 1, \\ m_k - \dfrac{1}{\pi} \displaystyle\int_{S'_d} s_k \log\left(\sum_{i=1}^{n} s_i^2\right) \boldsymbol{\Gamma}(d\mathbf{s}), & \textit{if} \quad \alpha = 1. \end{cases}$$

Proof. The proof is analogous to the proof of a similar result in the stable case. We refer the reader to [20] for details. ▮

Next, we show that all linear combinations of components of a multivariate ν-stable vector $\mathbf{Y}$ are univariate ν-stable.

Proposition 3.2. *Let* $\mathbf{Y} = (Y_1, ..., Y_d) \sim \nu_\alpha(\boldsymbol{\Gamma}, \mathbf{m})$ *be a* ν*-stable* (*respectively, strictly* ν*-stable, symmetric* ν*-stable*) *vector in* R^d *with ch.f. given in* (5)–(6). *Then any linear combination of the components of* $\mathbf{Y}$, $Y_{\mathbf{b}} = \sum_{k=1}^{d} b_k Y_k$, *is a univariate* ν*-stable* (*respectively, strictly* ν*-stable, symmetric* ν*-stable*) *random variable* $\nu_\alpha(\sigma, \beta, m)$ *given by* (8), *where*

$$\sigma = \left[\int_{S_d} |\langle \mathbf{b}, \mathbf{s}\rangle|^\alpha \boldsymbol{\Gamma}(d\mathbf{s})\right]^{1/\alpha}, \qquad \beta = \frac{\int_{S_d} |\langle \mathbf{b}, \mathbf{s}\rangle|^\alpha \, sign(\langle \mathbf{b}, \mathbf{s}\rangle) \, \boldsymbol{\Gamma}(d\mathbf{s})}{\int_{S_d} |\langle \mathbf{b}, \mathbf{s}\rangle|^\alpha \boldsymbol{\Gamma}(d\mathbf{s})},$$

$$m = \begin{cases} \langle \mathbf{b}, \mathbf{m}\rangle, & \alpha \neq 1, \\ \langle \mathbf{b}, \mathbf{m}\rangle - \dfrac{2}{\pi} \displaystyle\int_{S_d} \langle \mathbf{b}, \mathbf{s}\rangle \log |\langle \mathbf{b}, \mathbf{s}\rangle| \, \boldsymbol{\Gamma}(d\mathbf{s}), & \alpha = 1, \end{cases}$$

with the understanding that $\beta = 0$ *if* $\sigma = 0$.

Proof. The result follows from Theorem 3.1 and the corresponding result for stable laws (see [20]). ▮

Our next result shows that for a diagonal $d \times d$ real matrix D, the distribution of $D\mathbf{Y}$ is ν-stable if $\mathbf{Y} \sim \nu_\alpha(\boldsymbol{\Gamma}, \mathbf{m})$.

PROPOSITION 3.3. *Let* $\mathbf{Y} \sim \nu_\alpha(\mathbf{\Gamma}, \mathbf{m})$ *and let* $\mathbf{Y}_D = D\mathbf{Y}$, *where* $D = \{d_{ii}\}$ *is a diagonal matrix with all diagonal entries different from zero. Then,* $\mathbf{Y}_D \sim \nu_\alpha(\mathbf{\Gamma}_D, \mathbf{m}_D)$, *where*

$$\mathbf{m}_D = \begin{cases} D\mathbf{m}, & \text{if} \quad \alpha \neq 1, \\ D\mathbf{m} - 2/\pi \int_{S_d} D\mathbf{s} \log \|D\mathbf{s}\| \, \mathbf{\Gamma}(d\mathbf{s}), & \text{if} \quad \alpha = 1. \end{cases} \tag{14}$$

The spectral measure $\mathbf{\Gamma}_D$ *is defined as follows:*

$$\mathbf{\Gamma}_D(d\mathbf{s}) = \|D\mathbf{s}\|^\alpha \, \mathbf{\Gamma}'(d\mathbf{s}) \qquad \text{where} \quad \mathbf{\Gamma}'(B) = \mathbf{\Gamma}\left\{\mathbf{s}\colon \frac{D\mathbf{s}}{\|D\mathbf{s}\|} \in B\right\}.$$

Proof. Write $Ee^{i\langle \mathbf{t}, \mathbf{Y}_D\rangle} = Ee^{i\langle D\mathbf{t}, \mathbf{Y}\rangle} = \mathbf{\Psi}(D\mathbf{t})$, where $\mathbf{\Psi}$ is the ch.f. of $\mathbf{Y}$ given by (5)–(6), and apply the change of variable formula. ∎

Our final result shows that if $\mathbf{Y} \sim \nu_\alpha(\mathbf{\Gamma}, \mathbf{m})$ with discrete spectral measure, then all linear combinations of components of $\mathbf{Y}$ are jointly ν-stable.

PROPOSITION 3.4. *Let* $\mathbf{Y} = (Y_1, ..., Y_d) \sim \nu_\alpha(\mathbf{\Gamma}, \mathbf{m})$, *where* $\mathbf{\Gamma}$ *is concentrated on a finite number of points on the unit sphere* S_d. *Let* B *be an* $l \times d$ *real matrix. Then,* $\mathbf{Y}' = B\mathbf{Y}$ *is* ν*-stable.*

Proof. Let $\alpha \neq 1$. By Theorem 3.1, $\mathbf{Y} \stackrel{d}{=} \nu\mathbf{m} + \nu^{1/\alpha}\mathbf{X}$, where $\mathbf{X} \sim S_\alpha(\mathbf{\Gamma}, \mathbf{0})$. Since $\mathbf{\Gamma}$ is concentrated on a finite number of points of S_d, we have $\mathbf{X} \stackrel{d}{=} A\mathbf{X}'$, where A is a $d \times n$ real matrix and $\mathbf{X}' = (X'_1, ..., X'_n) \sim S_\alpha(\mathbf{\Gamma}, \mathbf{m}')$ has independent marginals (see, [20, Proposition 2.3.7]). Since $\mathbf{X}'' = BA\mathbf{X}'$ is stable (see, e.g., [20], Example 2.3.6), then $B\mathbf{Y} \stackrel{d}{=} \nu(B\mathbf{m}) + \nu^{1/\alpha}(BA\mathbf{X}')$ has the representation (10), and so it is ν-stable. The proof for $\alpha = 1$ is similar. ∎

4. TAIL PROBABILITIES

In this section we study tail probabilities of non-degenerate ν-stable random vectors in R^d. The tail behavior of their one-dimensional coordinates follows from Propositions 4.1 and 4.2 from [13] and Proposition 3.2 of the last section. Essentially, we have $P(Y_k > \lambda) = O(\lambda^{-\alpha})$ as $\lambda \to \infty$, where Y_k is the kth component of a ν-stable r.v. $\mathbf{Y}$ (the asymptotic behavior of the left tail of Y_k is the same). We show that the order statistics of $\mathbf{Y} = (Y_1, ..., Y_d) \sim \nu_\alpha(\mathbf{\Gamma}, \mathbf{m})$ (and their absolute values), as well as $Y = \|\mathbf{Y}\|$, have similar asymptotic behavior. We use the notation:

$$x_+ = \begin{cases} x, & \text{if} \quad x \geq 0, \\ 0, & x < 0. \end{cases} \tag{15}$$

In addition, we define

$$C_\alpha = \begin{cases} \dfrac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}, & \text{if} \quad \alpha \neq 1, \\ 2/\pi, & \text{if} \quad \alpha = 1. \end{cases} \tag{16}$$

THEOREM 4.1. *Let* $\mathbf{Y} = (Y_1, ..., Y_d) \sim \nu_\alpha(\mathbf{\Gamma}, \mathbf{m})$ *with* $0 < \alpha < 2$. *If* $\mathbf{Y}$ *is either*

(i) *strictly* ν*-stable* ($\alpha \neq 1$ *and* $\mathbf{m} = \mathbf{0}$ *or* $\alpha = 1$ *with* $\mathbf{g} = \mathbf{0}$) *and* $\eta = E\nu < \infty$

or

(ii) *non-strictly* ν*-stable and* $E\nu^{1 \vee \alpha} < \infty$ *for* $\alpha \neq 1$, *or* $E\,|\nu \log \nu| < \infty$ *for* $\alpha = 1$ *then*

$$\lim_{\lambda \to \infty} \lambda^\alpha P(\min_{1 \leqslant i \leqslant d} Y_i > \lambda) = C_\alpha \eta \int_{S_d} \min_{1 \leqslant i \leqslant d} [s_i]_+^\alpha \, \mathbf{\Gamma}(d\mathbf{s}). \tag{17}$$

Proof. We write the corresponding result for the stable case, extend it to the strictly ν-stable case, and then show that it holds in the general case as well via Lemma 4.4.2 of [20]. We sketch the proof for $0 < \alpha < 1$, as other cases are similar.

(i) Let $\mathbf{Y}$ be strictly ν-stable (with $0 < \alpha < 1$). By Theorem 3.1, we have $\mathbf{Y} = \nu^{1/\alpha}\mathbf{X}$, where $\mathbf{X} = (X_1, ..., X_d)$ is an ordinary stable vector with spectral measure $\mathbf{\Gamma}$. By [20, Theorem 4.4.1 and Remark 1, p. 188], we have

$$\lim_{\lambda \to \infty} \lambda^\alpha P(\min_{1 \leqslant i \leqslant d} X_i > \lambda) = C_\alpha \int_{S_d} \min_{1 \leqslant i \leqslant d} [s_i]_+^\alpha \, \mathbf{\Gamma}(d\mathbf{s}). \tag{18}$$

Conditioning on ν produces

$$\lim_{\lambda \to \infty} \lambda^\alpha P(\min_{1 \leqslant i \leqslant d} Y_i > \lambda) = \lim_{\lambda \to \infty} \int_0^\infty P(\min_{1 \leqslant i \leqslant d} X_i > \lambda z^{-1/\alpha})\, \lambda^\alpha \, dA(z). \tag{19}$$

Since the limit in (18) is finite, the integrand in (19) is bounded by Mz for some $M > 0$. Further, since $\int_0^\infty Mz\, dA(z) < \infty$, the dominated convergence theorem applied to the limit in (19), coupled with (18), produce (17). The result holds for the strictly ν-stable case.

(ii) Assume that $0 < \alpha < 1$ and $\eta = E\nu^{1 \vee \alpha} = E\nu < \infty$. By Theorem 3.1, we have $\mathbf{Y} \stackrel{d}{=} \mathbf{m}\nu + \nu^{1/\alpha}\mathbf{X}$, where $\mathbf{X} \sim S_\alpha(\mathbf{\Gamma}, \mathbf{0})$, $\nu \sim A$, and $\mathbf{X}$ and ν are independent. By part (i) of Theorem 4.1,

$$\lim_{\lambda \to \infty} \lambda^\alpha P(\min_{1 \leqslant i \leqslant d} \nu^{1/\alpha} X_i > \lambda) = C_\alpha \eta \int_{S_d} \min_{1 \leqslant i \leqslant d} [s_i]_+^\alpha \, \mathbf{\Gamma}(d\mathbf{s}). \tag{20}$$

If the limit (20) equals zero, we have

$$0 \leqslant \lim_{\lambda \to \infty} \lambda^\alpha P(\min_{1 \leqslant i \leqslant d} Y_i > \lambda)$$

$$\leqslant \lim_{\lambda \to \infty} \lambda^\alpha P(v \max_{1 \leqslant i \leqslant d} m_i > \lambda/2) + \lim_{\lambda \to \infty} \lambda^\alpha P(\min_{1 \leqslant i \leqslant d} v^{1/\alpha} X_i > \lambda/2) = 0,$$

since $\lambda^\alpha P(v \max_{1 \leqslant i \leqslant d} m_i > \lambda/2)$ either equals zero (if $\max_{1 \leqslant i \leqslant d} m_i < 0$) or converges to zero (if $\max_{1 \leqslant i \leqslant d} m_i \geqslant 0$, as $Ev < \infty$). Next, assume that the limit (20) is not zero, and note that

$$\min_{1 \leqslant i \leqslant d} v^{1/\alpha} X_i - W \leqslant \min_{1 \leqslant i \leqslant d} Y_i \leqslant W + \min_{1 \leqslant i \leqslant d} v^{1/\alpha} X_i, \tag{21}$$

where $W = v \max_{1 \leqslant i \leqslant d} |m_i|$ is a positive random variable. By (20), the random variable $X = \min_{1 \leqslant i \leqslant d} v^{1/\alpha} X_i$ has regularly varying tail, and the tail of X dominates the tail of W in the sense that $\lim_{\lambda \to \infty} P(W > \lambda)/P(X > \lambda) = 0$. The application of Lemma 4.4.2 of [20] now produces

$$\lim_{\lambda \to \infty} \frac{P(\min_{1 \leqslant i \leqslant d} Y_i > \lambda)}{P(\min_{1 \leqslant i \leqslant d} v^{1/\alpha} X_i > \lambda)} = 1,$$

and the result follows. Cases $\alpha = 1$ and $1 < \alpha < 2$ are similar. ∎

Corollary 4.1. *Under the conditions of Theorem* 4.1,

$$\lim_{\lambda \to \infty} \lambda^\alpha P(\min_{1 \leqslant i \leqslant d} |Y_i| > \lambda) = C_\alpha \eta \int_{S_d} \min_{1 \leqslant i \leqslant d} |s_i|^\alpha \, \mathbf{\Gamma}(d\mathbf{s}).$$

Proof. Let $\Delta = \{(\delta_1, ..., \delta_d): \delta_k = \pm 1, k = 1, ..., d\}$. Write

$$\lambda^\alpha P(\min_{1 \leqslant i \leqslant d} |Y_i| > \lambda) = \sum_{(\delta_1, ..., \delta_d) \in \Delta} \lambda^\alpha P(\min_{1 \leqslant i \leqslant d} (\delta_i Y_i) > \lambda) \tag{22}$$

and note that, by Proposition 3.3, $(\delta_1 Y_1, ..., \delta_d Y_d)$ is v-stable with the spectral measure $\mathbf{\Gamma}'(B) = \mathbf{\Gamma}(\mathbf{s}: \delta_1 s_1 + \cdots + \delta_d s_d \in B)$. Thus, by Theorem 4.1 and the change of variable formula, we obtain

$$\lim_{\lambda \to \infty} \lambda^\alpha P(\min_{1 \leqslant i \leqslant d} (\delta_i Y_i) > \lambda) = C_\alpha \eta \int_{S_d} \min_{1 \leqslant i \leqslant d} [s_i]_+^\alpha \, \mathbf{\Gamma}'(d\mathbf{s})$$

$$= C_\alpha \eta \int_{S_d} \min_{1 \leqslant i \leqslant d} [\delta_i s_i]_+^\alpha \, \mathbf{\Gamma}(d\mathbf{s}). \tag{23}$$

Next, equations (22) and (23) produce

$$\lim_{\lambda \to \infty} \lambda^\alpha P(\min_{1 \leqslant i \leqslant d} |Y_i| > \lambda) = C_\alpha \eta \int_{S_d} \sum_{(\delta_1, ..., \delta_d) \in \Delta} \min_{1 \leqslant i \leqslant d} [\delta_i s_i]_+^\alpha \, \mathbf{\Gamma}(d\mathbf{s}). \tag{24}$$

Finally, note that for fixed $\mathbf{s} = (s_1, ..., s_d) \in S_d$ the integrand in (24) is non-zero only if $s_i \neq 0$ for all $i = 1, ..., d$, in which case it is equal to $\min_{1 \leqslant i \leqslant d} |s_i|^\alpha$. ∎

To describe the asymptotic behavior of general order statistics, we need the following notation. Let $\{a_1, ..., a_d\}$ be a set of real numbers. Then, $a_{(1)} \geqslant a_{(2)} \geqslant \cdots \geqslant a_{(d)}$ denotes a non-increasing permutation of $\{a_1, ..., a_d\}$, while $|a|_{(1)} \geqslant |a|_{(2)} \geqslant \cdots \geqslant |a|_{(d)}$ denotes a non-increasing permutation of $\{|a_1|, ..., |a_d|\}$.

THEOREM 4.2. *Let* $\mathbf{Y} = (Y_1, ..., Y_d) \sim \nu_\alpha(\mathbf{\Gamma}, \mathbf{m})$. *If the conditions of Theorem* 4.1 *hold, the for any* $k = 1, ..., d$,

$$\lim_{\lambda \to \infty} \lambda^\alpha P(Y_{(k)} > \lambda) = C_\alpha \eta \int_{S_d} [s_{(k)}]_+^\alpha \, \mathbf{\Gamma}(d\mathbf{s}) \tag{25}$$

and

$$\lim_{\lambda \to \infty} \lambda^\alpha P(|Y|_{(k)} > \lambda) = C_\alpha \eta \int_{S_d} |s_{(k)}|^\alpha \, \mathbf{\Gamma}(d\mathbf{s}), \tag{26}$$

where C_α *is given in* (16) *while* $[s_{(k)}]_+^\alpha$ *and* $|s_{(k)}|^\alpha$ *are the* k*th largest among* $[s_i]_+^\alpha$ *and* $|s_i|^\alpha$, $i = 1, ..., d$, *respectively.*

Proof. The proof is similar to that of the corresponding result in the stable case, and thus omitted. See [20] for details. ∎

In conclusion, we compute the rate of decay for the norm of a ν-stable vector.

THEOREM 4.3. *Let* $\mathbf{Y} = (Y_1, ..., Y_d) \sim \nu_\alpha(\mathbf{\Gamma}, \mathbf{m})$ *with representation* (10) *and* $0 < \alpha < 2$. *If* $\mathbf{Y}$ *is either*

(i) *strictly* ν-*stable with* $\eta = E\nu < \infty$, *or*

(ii) *general* ν-*stable with* $E\nu^{1 \vee \alpha} < \infty$ *for* $\alpha \neq 1$ *or* $E|\nu \log \nu| < \infty$ *for* $\alpha = 1$ *then for any Borel set* $B \subseteq S_d$ *such that* $\mathbf{\Gamma}(\partial B) = 0$ *we have*

$$\lim_{\lambda \to \infty} \lambda^\alpha P(\|\mathbf{Y}\| > \lambda, \mathbf{Y}/\|\mathbf{Y}\| \in B) = C_\alpha \mathbf{\Gamma}(B) \, \eta, \tag{27}$$

with C_α *as in* (16).

Proof. The proof for strictly ν-stable case is similar to that of Theorem 4.1. By Theorem 3.1, $\mathbf{Y} \stackrel{d}{=} \nu^{1/\alpha}\mathbf{X}$, where $\mathbf{X} \sim S_\alpha(\mathbf{\Gamma}, \mathbf{m})$ with $\mathbf{m} = \mathbf{0}$ for $\alpha \neq 1$. Write

$$\lim_{\lambda \to \infty} \lambda^\alpha P(\|\mathbf{Y}\| > \lambda, \mathbf{Y}/\|\mathbf{Y}\| \in B) = \lim_{\lambda \to \infty} \int_0^\infty f(\lambda z^{-1/\alpha})\, z\, dA(z), \tag{28}$$

where $f(y) = y^\alpha P(\|\mathbf{X}\| > y, \mathbf{X}/\|\mathbf{X}\| \in B)$. By [20, Theorem 4.4.8],

$$\lim_{y \to \infty} f(y) = C_\alpha \mathbf{\Gamma}(B), \tag{29}$$

so that f is bounded. Thus, the integrand in (28) is bounded by an integrable function (as $\eta = E\nu < \infty$), and the result follows by the dominated convergence theorem.

We now turn to the non-strictly ν-stable case. We proceed by showing the following two statements:

$$\limsup_{\lambda \to \infty} \lambda^\alpha P(\|\mathbf{Y}\| > \lambda, \mathbf{Y}/\|\mathbf{Y}\| \in B) \leqslant C_\alpha \mathbf{\Gamma}(B)\, \eta \tag{30}$$

for all closed Borel sets $B \subseteq S_d$, and

$$\lim_{\lambda \to \infty} \lambda^\alpha P(\|\mathbf{Y}\| > \lambda) = C_\alpha \mathbf{\Gamma}(S_d)\, \eta. \tag{31}$$

If (30) and (31) hold then, as $\lambda \to \infty$, the measures P_λ defined by $P_\lambda(B) = \lambda^\alpha P(\|\mathbf{Y}\| > \lambda, \mathbf{Y}/\|\mathbf{Y}\| \in B)$ converge vaguely to the measure $C_\alpha \mathbf{\Gamma}$. If (31) holds, then they also converge weakly, which proves (27).

Case $\alpha \neq 1$. We start with (30) and follow the method of the proof of theorem 4.4.8 in [20].

By Theorem 3.1 we have $\mathbf{Y} \stackrel{d}{=} \nu\mathbf{m} + \nu^{1/\alpha}\mathbf{X}$ with $\mathbf{X} \sim S_\alpha(\mathbf{\Gamma}, \mathbf{0})$. For any $\varepsilon \in (0, 0.5)$,

$$\begin{aligned} &P\left(\|\mathbf{Y}\| > \lambda, \frac{\mathbf{Y}}{\|\mathbf{Y}\|} \in b\right) \\ &\quad \leqslant P\left(\|\nu\mathbf{m}\| \leqslant \varepsilon\lambda, \|\nu^{1/\alpha}\mathbf{X}\| > (1-\varepsilon)\lambda, \frac{\mathbf{Y}}{\|\mathbf{Y}\|} \in B\right) + P(\|\nu\mathbf{m}\| > \varepsilon\lambda). \end{aligned} \tag{32}$$

Since $E\nu^{1 \vee \alpha} < \infty$, we have

$$\lim_{\lambda \to \infty} \lambda^\alpha P(\|\nu\mathbf{m}\| \geqslant \varepsilon\lambda) = \lim_{\lambda \to \infty} \lambda^\alpha P(\nu^\alpha \geqslant (\varepsilon\lambda/\|\mathbf{m}\|)^\alpha) = 0. \tag{33}$$

Suppose that the set B is closed, and for any $\delta > 0$ let B_δ denote the closed δ-neighborhood of B. Since

$$\left\| \frac{\mathbf{Y}}{\|\mathbf{Y}\|} - \frac{\nu^{1/\alpha}\mathbf{X}}{\|\nu^{1/\alpha}\mathbf{X}\|} \right\| \leqslant \frac{2\,\|\nu\mathbf{m}\|}{\|\mathbf{Y}\|},$$

then

$$P\left(\|\nu\mathbf{m}\| \leqslant \varepsilon\lambda,\ \|\nu^{1/\alpha}\mathbf{X}\| > (1-\varepsilon)\,\lambda,\ \frac{\mathbf{Y}}{\|\mathbf{Y}\|} \in B,\ \frac{\nu^{1/\alpha}\mathbf{X}}{\|\nu^{1/\alpha}\mathbf{X}\|} \notin B_{\varepsilon'} \right) = 0,$$

where $\varepsilon' = 2\varepsilon/(1-2\varepsilon)$. Thus,

$$\begin{aligned} &P\left(\|\nu\mathbf{m}\| \leqslant \varepsilon\lambda,\ \|\nu^{1/\alpha}\mathbf{X}\| > (1-\varepsilon)\,\lambda,\ \frac{\mathbf{Y}}{\|\mathbf{Y}\|} \in B \right) \\ &\quad \leqslant P\left(\|\nu^{1/\alpha}\mathbf{X}\| > (1-\varepsilon)\,\lambda,\ \frac{\nu^{1/\alpha}\mathbf{X}}{\|\nu^{1/\alpha}\mathbf{X}\|} \in B_{\varepsilon'} \right). \end{aligned} \tag{34}$$

Using (32), (33) and (34), and applying the result for the strictly ν-stable vector $\nu^{1/\alpha}\mathbf{X}$, we have

$$\begin{aligned} &\limsup_{\lambda\to\infty} \lambda^\alpha P\left(\|\mathbf{Y}\| > \lambda,\ \frac{\mathbf{Y}}{\|\mathbf{Y}\|} \in B \right) \\ &\quad \leqslant \limsup_{\lambda\to\infty} \lambda^\alpha P\left(\|\nu^{1/\alpha}\mathbf{X}\| > (1-\varepsilon)\,\lambda,\ \frac{\nu^{1/\alpha}\mathbf{X}}{\|\nu^{1/\alpha}\mathbf{X}\|} \in B_{\varepsilon'} \right) \\ &\quad = \limsup_{\lambda\to\infty} [(1-\varepsilon)\,\lambda]^\alpha\,(1-\varepsilon)^{-\alpha}\,P\left(\|\nu^{1/\alpha}\mathbf{X}\| > (1-\varepsilon)\,\lambda,\ \frac{\nu^{1/\alpha}\mathbf{X}}{\|\nu^{1/\alpha}\mathbf{X}\|} \in B_{\varepsilon'} \right) \\ &\quad \leqslant (1-\varepsilon)^{-\alpha}\, C_\alpha\, \boldsymbol{\Gamma}(B_{\varepsilon'})\,\eta. \end{aligned} \tag{35}$$

When $\varepsilon \downarrow 0$, we have $B_{\varepsilon'} \downarrow B$ and (30) follows.

We turn to the proof of (31). Set $X = \|\nu^{1/\alpha}\mathbf{X}\|$ and $X = \|\nu\mathbf{m}\|$ and note that

$$P(X - W > \lambda) \leqslant P(\|\mathbf{Y}\| > \lambda) \leqslant P(X + W > \lambda). \tag{36}$$

Since X is strictly ν-stable and $E\nu^{1\vee\alpha} < \infty$, we conclude that X has a regularly varying tail, that dominates the tail of W. Thus, by Lemma 4.4.2 of [20] and (36),

$$\lim_{\lambda\to\infty} P(\|\mathbf{Y}\| > \lambda)/P(X > \lambda) = 1,$$

and the first part of the Theorem applied to X gives (31). This concludes the proof for $\alpha \neq 1$.

Case $\alpha = 1$. Write $\mathbf{Y} \stackrel{d}{=} 2/\pi\nu \log \nu \mathbf{g} + \nu(\mathbf{X} + \mathbf{m})$ and proceed as before using the first part of the Theorem applied to the strictly ν-stable random vector $\nu(\mathbf{X} + \mathbf{m})$. We omit the details. ∎

5. JOINT MOMENTS

We give necessary and sufficient conditions for finiteness of the joint moments of a non-degenerate $\mathbf{Y} = (Y_1, ..., Y_d) \sim \nu_\alpha(\mathbf{\Gamma}, \mathbf{m})$, whose components are n-fold dependent, that is

$$\mathbf{\Gamma}\{\mathbf{s} = (s_1, ..., s_d) \in S^d \colon s_1 \neq 0, s_2 \neq 0, ..., s_d \neq 0\} > 0.$$

THEOREM 5.1. *Let* $(Y_1, ..., Y_d) \sim \nu_\alpha(\mathbf{\Gamma}, \mathbf{m})$, *where* $0 < \alpha < 2$, *be jointly* ν*-stable and* n*-fold dependent, and let* $p_1, ..., p_d$ *be non-negative numbers.*

(i) *If* $\alpha \neq 1$, *then* $E\,|Y_1|^{p_1} \cdots |Y_d|^{p_d} < \infty$ *if and only if* $p = p_1 + \cdots + p_d < \alpha$ *and* $E\nu^{(p/\alpha) \vee p} < \infty$.

(ii) *If* $\alpha = 1$, *then* $E\,|Y_1|^{p_1} \cdots |Y_d|^{p_d} < \infty$ *if and only if* $p = p_1 + \cdots + p_d < \alpha$ *and* $E\nu^p(\log \nu)^{p'} < \infty$, *where* $p' = \sum_{i \in A} p_i$, *and* $A = \{i \colon g_i = \int_{S_d} s_i \mathbf{\Gamma}(d\mathbf{s}) \neq 0\}$ ($p' = 0$ *for* $A = \varnothing$).

Proof. We use representation Theorem 3.1 and the corresponding result for ordinary stable vectors [20, Lemma 4.5.2]. We prove the result when $\alpha < 1$, as the cases $\alpha = 1$ and $1 < \alpha < 2$ are similar.

Necessity. Assume that $E\,|Y_1|^{p_1} \cdots |Y_d|^{p_d} < \infty$. Then, there exists a positive constant z such that

$$E \prod_{i=1}^{d} |m_i z + z^{1/\alpha} X_i|^{p_i} < \infty, \tag{37}$$

and positive constants $x_1, ..., x_d$ such that

$$E \prod_{i=1}^{d} |m_i \nu + \nu^{1/\alpha} x_i|^{p_i} < \infty, \tag{38}$$

where ν and X_i's are as in the representation Theorem 3.1. By (37), we have $E \prod_{i=1}^{d} |X_i|^{p_i} < \infty$. Therefore, by Lemma 4.5.2 in [20], we must have $p < \alpha$. Turning to (38), we note that since $z^{1-1/\alpha} \to 0$ as $z \to \infty$, there exists a positive constant M such that if $z > M$ then $|m_i z^{1-1/\alpha} + x_i| \geqslant |x_i/2|$ for all $i = 1, ..., d$. Define $v_M = v \cdot I\{v > M\}$ and note that

$$\infty > E \prod_{i=1}^{d} |m_i v + v^{1/\alpha} x_i|^{p_i} \geqslant E \prod_{i=1}^{d} |m_i v_M + v_M^{1/\alpha} x_i|^{p_i}$$

$$= E \prod_{i=1}^{d} v_M^{p_i/\alpha} |m_i v_M^{1-1/\alpha} + x_i|^{p_i} \geqslant E v_M^{p/\alpha} \prod_{i=1}^{d} |x_i/2|^{p_i}.$$

Thus, $E v_M^{p/\alpha} < \infty$, so that $E v^{p/\alpha} < \infty$ and the result follows.

Sufficiency. Assume that $p < \alpha$ and $E v^{(p/\alpha) \vee p} = E v^{p/\alpha} < \infty$. By Hölder inequality,

$$E \prod_{i=1}^{d} |m_i v + v^{1/\alpha} X_i|^{p_i} \leqslant \prod_{i=1}^{d} (E |m_i v + v^{1/\alpha} X_i|^p)^{p_i/p}.$$

Note that since $0 < \alpha < 1$, we have $0 < z < z^{1/\alpha}$ if and only if $z > 1$. Conditioning on v, we obtain

$$\begin{aligned}
&E |m_i v + v^{1/\alpha} X_i|^p \\
&\quad = \int_0^\infty E |m_i z + z^{1/\alpha} X_i|^p \, dA(z) \\
&\quad \leqslant \int_0^\infty E(|m_i| \, z + z^{1/\alpha} \, |X_i|)^p \, dA(z) \\
&\quad = \int_0^1 E(|m_i| \, z + z^{1/\alpha} \, |X_i|)^p \, dA(z) + \int_1^\infty E(|m_i| \, z + z^{1/\alpha} \, |X_i|)^p \, dA(z) \\
&\quad \leqslant \int_0^1 z^p E(|m_i| + |X_i|)^p \, dA(z) + \int_1^\infty z^{p/\alpha} E(|m_i| + |X_i|)^p \, dA(z) \\
&\quad = E(|m_i| + |X_i|)^p \left(\int_0^1 z^p \, dA(z) + \int_1^\infty z^{p/\alpha} \, dA(z) \right) \\
&\quad = E(|m_i| + |X_i|)^p \, (I_1 + I_2) < \infty,
\end{aligned}$$

since $E(|m_i| + |X_i|)^p < \infty$ (as $p < \alpha$), $I_1 < \infty$ as an integral over a bounded set, and $I_2 \leqslant E v^{p/\alpha} < \infty$ by assumption. This completes the proof for $\alpha < 1$. ∎

ACKNOWLEDGMENTS

We thank the anonymous referees and the editors for helpful discussion and comments. The late Professor Stamatis Cambanis suggested the study of v-stable laws. His suggestions and comments led to this work.

REFERENCES

1. D. N. Anderson, A multivariate Linnik distribution, *Statist. Probab. Lett.* **14** (1992), 333–336.
2. D. N. Anderson and B. C. Arnold, Linnik distributions and processes, *J. Appl. Probab.* **30** (1993), 330–340.
3. J. Bertoin, "Lévy processes," Cambridge Univ. Press, Cambridge, UK, 1996.
4. J. Bunge, Composition semigroups and random stability, *Ann. Probab.* **24**(3) (1996), 1476–1489.
5. B. Erdogan and I. V. Ostrovskii, Analytic and asymptotic properties of generalized Linnik probability densities, *J. Math. Anal. Appl.* **217** (1998), 555–578.
6. B. W. Huff, The strict subordination of differential process, *Sankhya Ser. A* **31** (1969), 403–412.
7. V. Kalashnikov, "Geometric Sums: Bounds for Rare Events with Applications," Kluwer Academic, Dordrecht, 1997.
8. L. B. Klebanov, G. M. Manija, and J. A. Melamed, A problem of Zolotarev and analogs of infinitely divisible and stable distributions in a scheme for summing a random number of random variables, *Theory Probab. Appl.* **29** (1984), 791–794.
9. L. B. Klebanov, G. M. Manija, and J. A. Melamed, ν_p-strictly stable laws and estimation of their parameters, *Lecture Notes in Math.* **1233** (1987), 21–31.
10. L. B. Klebanov and S. T. Rachev, Sums of random number of random variables and their approximations with ν-accompanying infinitely divisible laws, *Serdica Math. J.* **22** (1996), 471–496.
11. T. J. Kozubowski, Characterization of multivariate geometric stable distributions, *Statist. Decisions* **15** (1997), 397–416.
12. T. J. Kozubowski and S. T. Rachev, The theory of geometric stable distributions and its use in modeling financial data, *European J. Oper. Res.* **74** (1994), 310–324.
13. T. J. Kozubowski and A. K. Panorska, On moments and tail behavior of ν-stable random variables, *Statist. Probab. Lett.* **1996** (29), 307–315.
14. T. J. Kozubowski and A. K. Panorska, Multivariate geometric stable distributions in financial applications, *Math. Comput. Modelling*, to appear.
15. T. J. Kozubowski, K. Podgórski, and G. Samorodnitsky, Tails of Lévy measure of geometric stable random variables, *Extremes*, submitted.
16. S. Mittnik and S. T. Rachev, Alternative multivariate stable distributions and their applications to financial modeling, *in* "Stable Processes and Related Topics" (S. Cambanis *et al.*, Eds.), pp. 107–119, Birkhauser, Boston, 1991.
17. S. Mittnik and S. T. Rachev, Modeling asset returns with alternative stable distributions, *Econometric Rev.* **12**(3) (1993), 261–330.
18. I. V. Ostrovskii, Analytic and asymptotic properties of multivariate Linnik's distribution, *Math. Phys. Anal. Geom.* **2** (1995), 436–455.
19. J. Rosiński, Weak compactness of laws of random sums of identically distributed random vectors in Banach spaces, *Coloq. Math.* **35** (1976), 313–325.
20. G. Samorodnitsky and M. Taqqu, "Stable Non-Gaussian Random Processes," Chapman & Hall, New York, 1994.
21. D. Szasz, On classes of limit distributions for sums of a random number of identically distributed independent random variables, *Theory Probab. Appl.* **17** (1972), 401–415.